Definition 7. *The analytic piecewise extension of the super-logarithm*

$$s(x,z)_n = -1 + \sum_{k=1}^{n} \frac{z^k}{k!} v_k(x) \qquad \text{if } x > 1,\ 0 < z \le 1$$

$$\text{where} \quad v_k(x) = \boldsymbol{D}_z^k [\, s(x, x^z)_n - 1\,]_{z=0}$$

$$\text{or} \quad v_k(x) = \left[\left\langle \left\langle \frac{m^k}{m!} - \delta_{mk} \log(x)^{-k} \right\rangle_{m=1}^{n} \right\rangle_{k=0}^{n-1} \right]^{-1} \left\langle \delta_{m1} \right\rangle_{m=1}^{n}$$

$$\operatorname{slog}_x(z)_n = s(x, x^z)_n - 1 \qquad \text{if } x > 1,\ z \le 0$$

$$\operatorname{slog}_x(z)_n = s(x, z)_n \qquad \text{if } x > 1,\ 0 < z \le 1$$

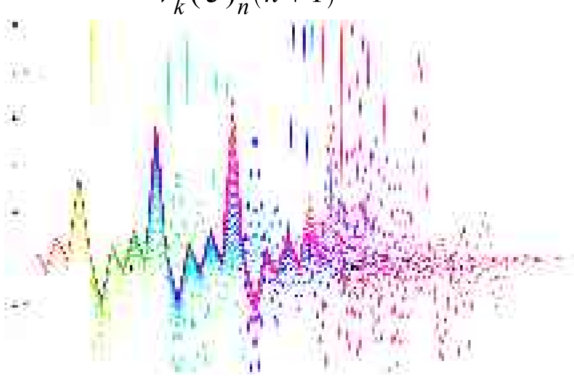
$$\operatorname{slog}_x(z)_n = s(x, \log_x^m(z))_n + m \qquad \text{if } x > 1,\ ({}^{m-1}x) < z \le ({}^{m}x),\ m > 0$$

$$\operatorname{slog}_x(z) = \lim_{n \to \infty} \operatorname{slog}_x(z)_n \qquad \text{if } x > 1$$

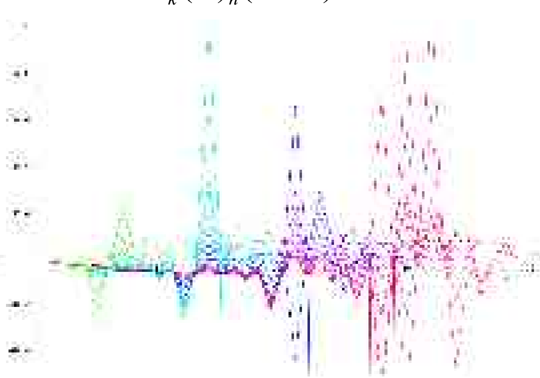
where the superscript (–1) indicates the inverse matrix.

We already know that the super-logarithm defined in this manner will satisfy property (2), because it is a piecewise extension. Although we can soon find if the super-logarithm satisfies property (3), it still is not obvious that the series converges, or that the final limit exists. To show this, we need to take a closer look at $v_k(x)$. Applying the ratio test for when $x = \boldsymbol{e}$, we get a rather chaotic graph. Looking closely at this graph, there is a pattern that repeats every seven terms, so we find that the graph of the ratios of the terms is much smoother when the ratio is for those terms that are seven terms apart:

graph of $\dfrac{v_{k+1}(\boldsymbol{e})_n}{v_k(\boldsymbol{e})_n (k+1)}$ in k from 1 to n



graph of $\dfrac{v_{k+7}(\boldsymbol{e})_n k!}{v_k(\boldsymbol{e})_n (k+7)!}$ in k from 1 to n



where the second graph seems to fall on a curve for low k. This curve can be expressed algebraically by the approximate recurrence relations:

$$v_{k+7}(\boldsymbol{e}) \approx -v_k(\boldsymbol{e})\boldsymbol{e}^{2\pi} k^{2.95} k^{\log\sqrt{k}}$$

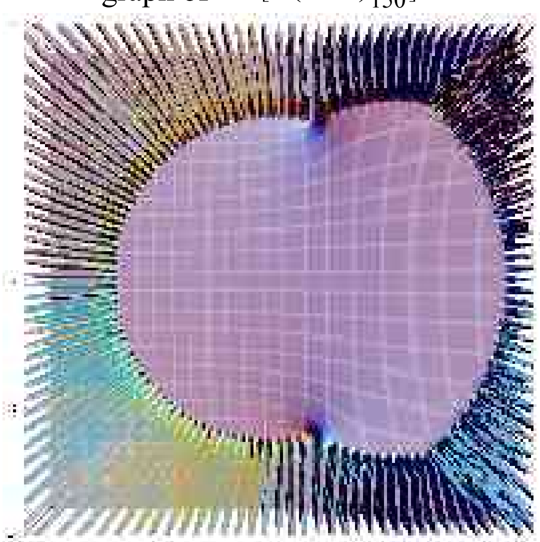
or: $$v_{k+7}(\boldsymbol{e}) \approx -v_k(\boldsymbol{e}) k^{7(7/k)^{1/k}}$$

where the magnitudes are roughly:

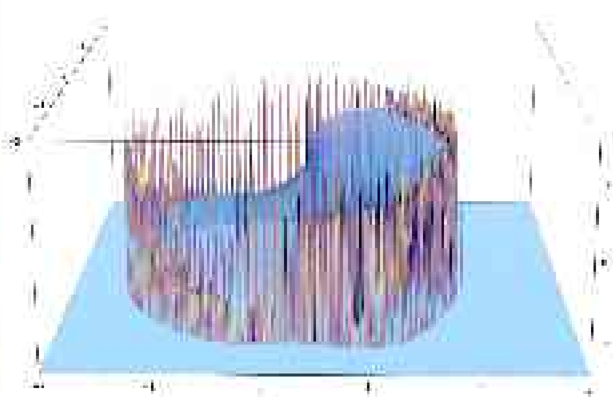
$$\left|v_k(\boldsymbol{e})\right| \approx k^{(k^2/2950 + k/\sqrt{2} - 3)}$$

which helps explain the signs of the terms. These do not help us determine the convergence of the series, because in the limit as $k \to \infty$ they become infinite. For now, we can only see whether the approximations of the super-logarithm constructed as in definition (7), converge to each other. In order to do this we will look only at the critical function $s(\boldsymbol{e}, z)_{150}$. The definition of this function is not limited to real numbers, it is only the piecewise-defined $\text{slog}_x(z)_n$ that is limited to real numbers. Outside a certain region in the complex plane, complex z will give extreme values that do not correspond to previous approximations, so we can identify an approximate radius of convergence by looking at some 3D plots of the *real part* of the critical function $s(\boldsymbol{e}, z)_{150}$:

graph of $\Re[s(\boldsymbol{e}, z)_{150}]$



graph of $\Re[s(\boldsymbol{e}, z)_{150}] = y$ if $y < 2$



as you can see, although we haven't found the radius of convergence algebraically, there does seem to be a radius of convergence, and it covers the required domain of the critical function: $0 \le z \le 1$. These graphs and the tables of numerical data in appendix C are the best proof of convergence that the author has found. Hopefully a more exact method will be found, but until then we will assume it converges. There is no indication that it would not converge, but that is not enough to prove it does.

Assuming $\lim_{n\to\infty} s(x,z)_n$ converges for all $0 \le z \le 1$, the following shows the uniqueness and properties of the super-logarithm in definition (7):

Rough Proof

The critical functions $s(x,z)_n$ converge for finite *n*, and $\lim_{n\to\infty} s(x,z)_n$ seems to converge within the approximate radius of convergence shown above. There is no question if the critical functions are $\boldsymbol{C}^{\infty}$. The piecewise-defined functions $\text{slog}_x(z)_n$ are $\boldsymbol{C}^{n-1}$, by lemma (1), so in the limit $\text{slog}_x(z)_\infty$ is $\boldsymbol{C}^{\infty}$. This proves that $\text{slog}_x(z)_\infty$ satisfies property (3), and since the piecewise-defined functions $\text{slog}_x(z)_n$ are defined using property (2), they must satisfy it. Since the determinant being zero implies $0 < x < 1$, by lemma (2), the logical complement would be that the determinant is nonzero if $x > 1$. The determinant being nonzero then implies there is one and only one solution to the system of equations the matrix represents. So if $x > 1$, $\text{slog}_x(z)_\infty$ exists and is unique. Therefore, given an $x > 1$, $\text{slog}_x(z) = \text{slog}_x(z)_\infty$ exists, is unique, and satisfies properties (2) and (3). This proves that definition (7) is *the* analytic piecewise extension of the super-logarithm to all real number *z*. ∎

Generalization

Solving for the super-logarithm has produced “nice” $\boldsymbol{C}^n$ approximations. We can generalize this by going back to Abel's equation for linearizing a function:

$$\phi(f(z)) = \phi(z) + 1$$
$$\phi(z) = \phi(f(z)) - 1$$
$$\boldsymbol{D}^k[\phi(z)] = \boldsymbol{D}^k[\phi(f(z)) - 1]$$
$$\boldsymbol{D}^k[\phi(z)]_{z=0} = \boldsymbol{D}^k[\phi(f(z)) - 1]_{z=0}$$

$$\phi(z)_n = \sum_{k=0}^{n} \frac{z^k}{k!} \boldsymbol{D}^k[\phi(f(z)) - 1]_{z=0}$$
$$\phi(z) = \lim_{n\to\infty} \phi(z)_n$$

defining the linearizing function by the property we desire of it, and solving for approximations of it for some $0 \le k < n$. It is possible this could work for any analytic $f(z)$, but it is likely that additional conditions apply. If the equations have a solution, and the solutions converge, then any iterate of the function can be expressed as:

$$\phi(f(z)) = \phi(z) + 1$$
$$\phi(f^n(z)) = \phi(z) + n$$
$$f^n(z) = \phi^{-1}(\phi(z)+n)$$

This method is not restricted to continuously iterating a function over the real numbers, the series expansion should work within some complex radius of convergence.

Conclusion

We now have a unique analytic piecewise extension of the super-logarithm. Using the super-logarithm, we can define tetration as its inverse. Also, because the super-logarithm is the linearizing function of exponentials, we can use it to define continuously iterated exponentiation as well. Remember that the first definition of the super-logarithm requires that ${}^{\mathrm{slog}_x(a)}x = a = \exp_x^{\mathrm{slog}_x(a)}(1)$ hold in general. This allows the definition of continuously iterated exponentiation as follows:

Definition 8. *Tetration*

$$
\begin{aligned}
{}^{y}x &= \mathrm{slog}_x^{-1}(y) && \text{if } x > 1,\ y > -2 \\
{}^{y}x &= \log_x({}^{y+1}x) && \text{if } x > 1,\ y \le -2
\end{aligned}
$$

Definition 9. *Continuously iterated exponentiation*

$$\exp_x^y(a) = {}^{y+\mathrm{slog}_x(a)}x \quad \text{if } x > 1$$

In this paper, a unique analytic piecewise extension has been presented for the super-logarithm, tetration, and continuously iterated exponentiation. In appendix A, the method used for solving the systems of equations to produce the approximations of these extensions is given. Now that the numerical values of the super-logarithm and tetration can be found to any precision, it is only a matter of time before a closed-form definition of tetration and related functions is found.