Solving for the Analytic Piecewise Extension of Tetration and the Super-logarithm

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Abstract

An overview of previous extensions of tetration is presented. Specific conditions for differentiability and piecewise continuity are shown. This leads to a way of generating approximations of the super-logarithm. These approximations are shown to converge to a function that satisfies two basic properties of extensions of tetration.

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Introduction

Of the operators in the sequence: addition, multiplication, exponentiation, and tetration, only the first three are well-defined analytic operations. *Tetration*, also known as the hyper4 operator, power towers, and iterated exponentiation, has only been “nicely” defined for integer *y* in ${}^{y}x$. Although extensions of tetration to real *y* have been made, those extensions have not followed simply from the properties of hyper-operations. The particular form of this property as it pertains to tetration is:

Property 1. *Iterated exponential property*

$${}^{y}x = x^{({}^{y-1}x)} \quad \text{for all real } y.$$

Tetration has two inverses, the *super-root*, and the *super-logarithm*:

$$z = {}^{y}x = \operatorname{tet}_x(y) = \operatorname{twr}_y(x)$$
$$x = \operatorname{srt}_y(z) = \operatorname{twr}_y^{-1}(z)$$
$$y = \operatorname{slog}_x(z) = \operatorname{tet}_x^{-1}(z)$$

When *y* is fixed, the function $\operatorname{twr}_y(x)$ is called an *order y power tower of x* [14]. When *x* is fixed, the function $\operatorname{tet}_x(y)$ is called a *base x tetrational function of y*. In general, it is pronounced: *x tetra y*. Inverting a power tower gives the super-root, and inverting a tetrational function gives the super-logarithm, More detail about these inverses can be found in [12], [14] and [15]. The super-logarithm has an equivalent to property (1), and can be found by applying the super-logarithm to property (1) as follows:

$$y - 1 = \operatorname{slog}_x({}^{y-1}x)$$
$$y = \operatorname{slog}_x({}^{y}x) = \operatorname{slog}_x(x^{({}^{y-1}x)}) = \operatorname{slog}_x({}^{y-1}x) + 1$$

Replacing ${}^{y-1}x$ with *z*, we get the equivalent property for the super-logarithm:

Property 2. *Function linearization property*

$$\operatorname{slog}_x(x^z) = \operatorname{slog}_x(z) + 1$$

This property is actually a special case of Abel's equation for *linearizing* a function, which is found in [4] and [9]. Linearization is done in the hope of iterating a function continuously, or finding the *iterative roots* of a function [6]. The generalization $\phi(f(z)) = \phi(z)+1$ can be used to iterate a function continuously, as follows:

$$\begin{aligned}
&\phi(f(z)) = \phi(z) + 1\\
&\phi(f^n(z)) = \phi(z) + n\\
&f^n(z) = \phi^{-1}(\phi(z)+n)
\end{aligned}$$

where n could be real. In the case above, $f(z) = x^z$, and $\phi(z) = \mathrm{slog}_x(z)$. This means that if we can find the linearizing function of exponentials: $\mathrm{slog}_x(z)$, then not only can we define tetration as its inverse, but we can also use it to continuously iterate exponential functions.

The next property uses the notation $\boldsymbol{C}^n$. A function $f(x)$ is $\boldsymbol{C}^n$, if it is n-times differentiable. To be analytic, a function must be infinitely differentiable, or $\boldsymbol{C}^\infty$. All the other hyper-operations below tetration (hyper4) are $\boldsymbol{C}^\infty$ for both operands, so it is natural to require this of tetration as well. When combined with either property above, this property ensures a unique extension of tetration:

Property 3. *Infinite differentiability property*

$f(x)$ is $\boldsymbol{C}^\infty \equiv \boldsymbol{D}^k f(x)$ exists for all integer k.

Up to this point, extensions of tetration have been made that satisfy property (1), but not property (3), and extensions of tetration have been made that satisfy property (3), but property (1) only for integer y. To avoid discontinuous derivatives, and to be a valid extension of tetration or the super-logarithm, these properties are necessary. It is the goal of this paper, however, to show that these properties are sufficient to find such an extension, and that the extension found will be unique.

Background

History

Tetration has fascinated mathematicians for centuries, in part because it is as fundamental as addition or multiplication in its definition, yet was beyond the realm of calculation due to its fast growth rate. Although the calculation of tetration at low values such as ${}^{5}5$ is still a daunting task, computational advances have open the doors of solving large systems of equations involved in finding non-integer values.

Many hobbyist and professional mathematicians alike have rediscovered tetration, and asked themselves the questions: what is ${}^{0.5}2$ or ${}^{e}\boldsymbol{e}$ or ${}^{\pi}\boldsymbol{e}$? Is there an answer? If so, what is the derivative $\boldsymbol{D}_y({}^{y}x)$? These questions have motivated previous extensions of tetration, just as they have motivated this one. The advances made so far in tetration have not been by motivation alone. Many mathematicians have contributed to what is known about tetration today.

The first tetration-related proof was made by Euler, when he proved the convergence of ${}^{\infty}x$, although he used a different notation. The first person to publish the notation ${}^{y}x$ was Mauer [4], and was later popularized by Rucker in [13]. The word *tetration* was coined by Goodstein from the words *tetra* and *iteration* [10]. The sequence of operations from which tetration comes is generally known as the hyper-operations, or hyper-*n* operators. Goodstein also gives names to the hyper-operations: *pentation* for hyper5, *hexation* for hyper6, and so on. The first to write about this sequence of operations was Grzegorczyk [12], hence the sequence has also been called the Grzegorczyk hierarchy. When the hyper-operation sequence is viewed as a ternary function, it is sometimes called the Ackermann function [5]. Many people have introduced notations for this sequence of operators, and beyond, including Knuth [8], Conway [2], Munafo [11], and Bowers [1].

Euler's proof of the convergence of ${}^{\infty}x$, is useful in defining many other related functions, one of which is the Lambert *W* function, also known as the product-logarithm. There is also a relationship between these two functions and the second super-root. To overview the properties:

$$
\begin{aligned}
{}^{\infty}x &= u && \text{where } u = x^{u} \\
W(x) &= u && \text{where } u\boldsymbol{e}^{u} = x \\
\operatorname{srt}_2(x) &= u && \text{where } u^{u} = x
\end{aligned}
$$

and using only basic algebra, the following identities can be found:

$$
\begin{aligned}
{}^{\infty}x &= \frac{W(-\log(x))}{-\log(x)} && = 1/\operatorname{srt}_2(x^{-1}) \\
{}^{\infty}(\boldsymbol{e}^{-x})\,x &= W(x) && = x/\operatorname{srt}_2(\boldsymbol{e}^{x}) \\
1/{}^{\infty}(x^{-1}) &= \frac{\log(x)}{W(\log(x))} && = \operatorname{srt}_2(x)
\end{aligned}
$$

showing that these functions can all be expressed in terms of each other. So if it were a matter of choice, any would work, although many Computer Algebra Systems come with the product-logarithm. Aesthetically, though, ${}^{\infty}x$ seems a "nicer" choice.

Both the infinite power tower and the product logarithm have series expansions:

$$
{}^{\infty}x = \sum_{k=1}^{\infty} \frac{(k\log(x))^{k-1}}{k!}
$$

$$
W(x) = \sum_{k=0}^{\infty} \frac{(-k)^{k-1}x^{k}}{k!}
$$

and are very well-known, well-defined analytic functions.

Applications

The most immediate applications of tetration and the super-logarithm are in the representation of large numbers, and continuously iterated exponentiation. Some applications of the approximations given in this paper that may not be immediate are:

- Approximating continuous iterations of other functions
- Validating closed-form extensions of tetration or the super-logarithm
- Developing new algebraic identities using the approximations

One application of the super-logarithm could be in population modeling. In particular, a graph of the super-logarithm looks similar to a logistic population model. Instead of stopping at some point, though, the super-logarithm continues to grow.

Extensions

One of the reasons why extending ${}^{y}x$ to real y is so difficult, is that many of the laws that are instrumental in defining exponentiation (hyper3) do not hold for tetration (hyper4). For example, exponentiation satisfies $\sqrt[n]{x} = x^{1/n}$ whereas the same is not true of tetration; $\text{srt}_n(x) \neq {}^{1/n}x$. This is in part because there is no multiplicative law of exponents for tetration; ${}^{b}({}^{a}x) \neq {}^{(ab)}x$. Another aspect of exponentiation that is convenient is the change-of-base formula for logarithms. There is currently no known change-of-base formula for super-logarithms, however. The absence of these properties to fall back on means that a completely different method must be used to make progress.

Piecewise Extensions

A piecewise-defined extension of tetration uses property (1) to compute the values of tetration at all intervals, given the values in an interval of length one. So any piecewise extension of tetration must obey property (1) by definition. Also, one benefit of using a piecewise extension is that in coming up with extensions, the only part that needs to be taken into account is the *critical interval*, not the entire function. Before we get to some piecewise extensions lets define their general form:

Definition 1. *General piecewise extension of tetration*

$$
\begin{aligned}
{}^{y}x &= \log_x({}^{y+1}x) && \text{if } y \leq -1 \\
{}^{y}x &= t(x, y) && \text{if } -1 < y \leq 0 \\
{}^{y}x &= x^{({}^{y-1}x)} && \text{if } y > 0
\end{aligned}
$$

Definition 2. *General piecewise extension of the super-logarithm*

$$
\begin{aligned}
\text{slog}_x(z) &= s(x, x^z) - 1 && \text{if } z \leq 0 \\
\text{slog}_x(z) &= s(x, z) && \text{if } 0 < z \leq 1 \\
\text{slog}_x(z) &= s(x, \log_x^m(z)) + m && \text{if } ({}^{m-1}x) < z \leq ({}^{m}x),\ m > 0
\end{aligned}
$$

where $t(x, y)$ is the *critical function* of tetration, and $s(x, z)$ is the *critical function* of the super-logarithm. The most common extension, found in [4] and [15], is the extension that behaves like a line within the critical functions:

Extension 1. *Linear* $t(x, y)$

$$t(x, y) = y + 1 \quad \text{if} \quad -1 < y \leq 0$$

We can invert this definition to get a piecewise-defined extension of super-logarithms:

Extension 2. *Linear* $s(x, z)$

$$s(x, z) = z - 1 \quad \text{if} \quad 0 < z \leq 1$$

The reason why this can be done is that the value of tetration everywhere depends only on the value of tetration where $-1 < y \leq 0$, and the value of the super-logarithm everywhere depends only on the value of the super-logarithm where $0 < z \leq 1$. This will help when we find a series, because the only place we need convergence is between 0 and 1.

Analytic Extensions

An analytic extension of tetration or the super-logarithm is a function that satisfies property (3). This means that these functions can be represented as an infinite series expansion. Depending on the function, it may be preferable to expand it about different points or with respect to different variables. This is a list of the expansions used here:

Definition 3. *General series extension of tetration with respect to the hyper-base*

$${}^{y}x = \sum_{k=0}^{\infty} \log(x)^k \alpha_k(y)$$

Definition 4. *General series extension of tetration with respect to the hyper-exponent*

$${}^{y}x = \sum_{k=0}^{\infty} \frac{y^k}{k!} \beta_k(x)$$

Definition 5. *General series extension of the super-logarithm*

$$\operatorname{slog}_x(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!} v_k(x)$$

In [4], Galidakis gives an extension of the form in definition (3):

Extension 3. *Summary of Galidakis' analytic extension in* [4]

$$\begin{array}{ll} a_{m,n} = 1 & \text{if } m = n = 0 \\ a_{m,n} = 0 & \text{if } m = 0 \text{ and } n \neq 0 \\ a_{m,n} = \dfrac{1}{n!} & \text{if } m = 1 \\ a_{m,n} = \displaystyle\sum_{j=1}^{n} \frac{j}{n} a_{m,n-j}\, a_{m-1,j-1} & \text{otherwise} \end{array}$$

$$\begin{array}{ll} \phi(x) = \exp\left(\dfrac{4}{4x^2-1}\right) & \text{if } |x| < 1/2 \\ \phi(x) = 0 & \text{otherwise} \end{array}$$

$$\chi_m(x) = \frac{(a_{m,n} - a_{m-1,n})\,\phi(x-(m-1/2))}{\displaystyle\int_{m-1}^{m} \phi(t-(m-1/2))\,\boldsymbol{d}t}$$

$$\begin{array}{ll} \alpha_n(x) = 1 & \text{if } n = 0 \\ \alpha_n(x) = \displaystyle\int_0^x \sum_{m=1}^{n} \chi_m(t)\,\boldsymbol{d}t & \text{if } n \neq 0 \end{array}$$

Extension 4. *Analytic term function related to the piecewise linear critical function*

$$\begin{array}{ll} \alpha_k(y) = 1 & \text{if } k = 0,\ y > 1 \\ \alpha_k(y) = \dfrac{y^k}{k!} & \text{if } 0 \leq y \leq 1 \\ \alpha_k(y) = \displaystyle\sum_{j=1}^{k} \frac{j}{k} \alpha_{k-j}(y)\,\alpha_{j-1}(y-1) & \text{otherwise} \end{array}$$

which uses the recurrence relation found in [4].

To show that these two analytic extensions do not satisfy property (1), we can test values. If we find one real y for which property (1) is not satisfied, then it is not satisfied for all real y. According to Galidakis [4], extension (3) converges for all x in a compact subset of the complex plane, so the series converges for $x = \boldsymbol{e}$.

Using extension (3), the values: ${}^{0.5}\boldsymbol{e} = 1.858$ and ${}^{1.5}\boldsymbol{e} = 5.613$ do not satisfy property (1), because $\boldsymbol{e}^{1.858} = 6.413 \neq 5.613$. Doing the same using extension (4), the values: ${}^{0.5}\boldsymbol{e} = 1.649$ and ${}^{1.5}\boldsymbol{e} = 5.185$ do not satisfy property (1) because $\boldsymbol{e}^{1.649} = 5.200 \neq 5.185$.

Neither of these analytic extensions satisfy property (1), so neither of these are piecewise-definable. We could use then as critical functions of a piecewise definition of tetration, but doing so would cause them to fail to be analytic (property (3)).

Is there a way to find an extension that satisfies both properties? There is, but it is not an extension of tetration itself. It is an extension of the super-logarithm, the inverse of tetration, so by finding the values of a super-logarithm that satisfies property (2), and property (3), the inverse of the super-logarithm will satisfy property (1) and property (3).